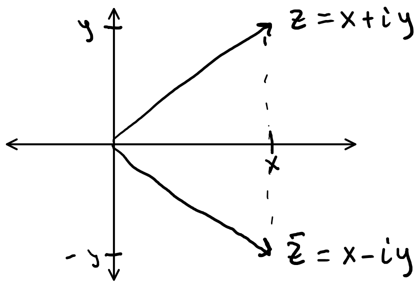
Definition (Complex Conjugate) Let $z = x+iy$ be a complex number. The **complex conjugate** of z is
$$\bar{z} = x - iy.$$

Geometrically, $\bar{z}$ is the reflection of z about the real axis.



Proposition (Properties of the conjugate) For all $z, w \in \mathbb{C}$:

(1) $\bar{\bar{z}} = z$

(2) $|\bar{z}| = |z|$

(3) $\overline{z+w} = \bar{z} + \bar{w}$

(4) $\overline{zw} = \bar{z}\,\bar{w}$

(5) $z\bar{z} = |z|^2$

(6) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

Proof. (1)–(3) Geometrically clear. (4), (6) Exercises.

(5) Let $z = x+iy$. Then
$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2 - ixy + iyx - i^2y^2 \\ &= x^2 + y^2 + i(yx - xy) \\ &= x^2 + y^2 \\ &= |z|^2. \end{aligned}$$

Notice that (5) gives a nice formula for z^{-1}. Suppose $z \neq 0$. Then by (5) $z\bar{z} = |z|^2 \Rightarrow z\dfrac{\bar{z}}{|z|^2} = 1$. Since inverses are unique,
$$\boxed{z^{-1} = \frac{\bar{z}}{|z|^2}.}$$

Recall that every nonzero point $(x,y) \in \mathbb{R}^2$ can be written in polar coordinates (r, θ) where

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

This suggests the following definition.

$\boxed{\text{Definition (Polar form)}}$ If (r,θ) are polar coordinates for (x,y), then the polar form of $z = x+iy$ is

$$z = r(\cos\theta + i\sin\theta).$$

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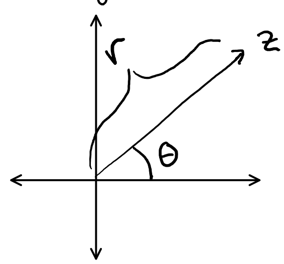
Evidently, r, θ are related to x, y by the equations

$$|z| = r \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

taking into account which quadrant (x,y) belongs to.



The value of θ is not unique. Each possible value is called an argument of z. The set of all possible arguments is denoted

$$\arg z.$$

The polar form is unique if we specify that $-\pi < \theta \leq \pi$. The unique argument in this interval is the principal argument $\operatorname{Arg} z$. Notice that

$$\arg z = \operatorname{Arg} z + 2k\pi \quad , \ k \in \mathbb{Z}$$

The polar form suggests a definition for the symbol $e^{i\theta}$:

$$e^{i\theta} \overset{\text{def}}{=} \cos\theta + i\sin\theta \qquad \text{(Euler's Formula)}$$

Then the polar form is written compactly in exponential form:

$$z = re^{i\theta}.$$

//

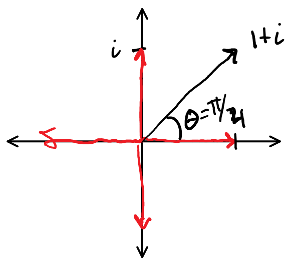
Example

(1) Exponential form of $1+i$: $|1+i| = \sqrt{2}$ and $\operatorname{Arg} 1+i = \pi/4$

So $1+i = \sqrt{2}\, e^{i\pi/4}$.

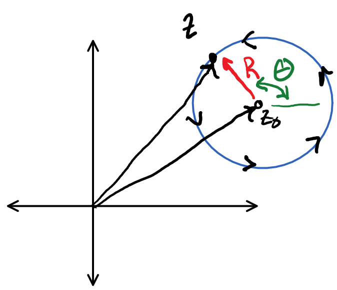
(2) $1 = e^{i0}$ $\quad i = e^{i\pi/2}$ $\quad -1 = e^{i\pi}$ $\quad -i = e^{-i\pi/2} = e^{i3\pi/2}$

Arg z $\qquad$ not Arg z



(3) The circle $C_R(z_0)$ has a nice parameterization:

$$z = z_0 + Re^{i\theta}, \quad 0 \le \theta \le 2\pi$$



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Proposition (Products / Powers in Exponential Form)

Let $z = re^{i\theta}$ and $w = se^{i\varphi}$. Then

(1) $zw = rs\, e^{i(\theta+\varphi)}$

(2) $z/w = \frac{r}{s} e^{i(\theta-\varphi)}$

(3) $z^{-1} = \frac{1}{r} e^{-i\theta}$

(4) $z^n = r^n e^{in\theta}$, $\forall n \in \mathbb{Z}$.

Proof. (1) $zw = (re^{i\theta})(se^{i\varphi})$

$$= rs(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$

$$= rs(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi))$$

$$= rs(\cos\theta+\varphi \;+ i\sin\theta+\varphi)$$
$$= rs\, e^{i(\theta+\varphi)}.$$

(4) Case 1: $n \geq 1$. Proof by induction. $n=1$ is clear. Let $n \geq 1$. Assume $z^n = r^n e^{in\theta}$. Then

$$z^{n+1} = z^n \cdot z$$
$$= r^n e^{in\theta} r e^{i\theta} \overset{(1)}{=} r^{n+1} e^{i(n\theta+\theta)} = r^{n+1} e^{i(n+1)\theta}.$$

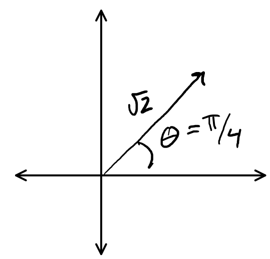
Case 2: $n<0$. Then put $m=-n$. Apply Case 1 w/

$$z^n = (z^{-1})^m$$

Case 3: $n=0$, by definition. ∎

$\boxed{\text{Example}}$

$$(1+i)^{2021} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{2021}$$
$$= \sqrt{2}^{2021} e^{i\left(\frac{2020\pi}{4}+\frac{\pi}{4}\right)}$$
$$= \sqrt{2}^{2020}\sqrt{2}\, e^{i\frac{2020\pi}{4}} e^{i\pi/4}$$
$$= 2^{1010} e^{505\pi} (1+i)$$
$$= -2^{1010}(1+i).$$



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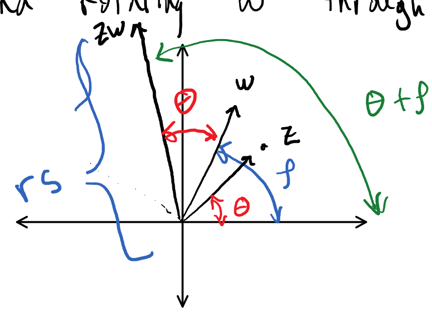
Part (1) of the proposition gives a geometric interpretation of complex multiplication. If $z = re^{i\theta}$ and $w = se^{i\varphi}$, then $zw = rs\,e^{i(\theta+\varphi)}$. This just says that zw is obtained from w by scaling w by a factor of $|z|=r$ and rotating w through an angle of $\operatorname{Arg} z$.



A couple more interesting consequences:

(1) the unit circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ is closed under multiplication.

(2) De Moivre's formula: from (4) w/ $z = e^{i\theta} \Rightarrow z^n = e^{in\theta}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

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$\boxed{\text{Proposition (Arguments of Products)}}$ Let z, w be nonzero complex, then

(1) $\arg zw = \arg z + \arg w$

(2) $\arg z/w = \arg z - \arg w$.

The statements are interpreted as follows: given the values of any 2 of the arguments, there is a value of the third satisfying the equation.

Proof. Let $z = re^{i\theta}$, $w = se^{i\varphi}$ (given values θ, φ of $\arg z / \arg w$). Then $\theta + \varphi$ is an argument of zw satisfying (1), by (4) of the proposition. Say we are given a value of $\arg zw$. Then it must be of the form

$$(\theta + \varphi) + 2k_1\pi \quad \text{for some } k_1 \in \mathbb{Z}.$$

If we are given a value of $\arg z$, it must be of the form

$$\theta + 2k_2\pi \quad \text{for some } k_2 \in \mathbb{Z}.$$

☆ We need to find a value τ of $\arg w$ such that

$$(\theta + \varphi) + 2k_1\pi = (\theta + 2k_2\pi) + \tau.$$

Lets take $\tau = \varphi + 2(k_1 - k_2)\pi$, which is an argument of z.

Then
$$(\theta + 2k_2\pi) + \tau = \theta + 2k_2\pi + \varphi + 2(k_1 - k_2)\pi$$
$$= \theta + \varphi + 2k_1\pi$$

as desired. This proves (1). Part (2) follows from (1):

$\arg \frac{z}{w} = \arg zw^{-1} = \arg z + \arg w^{-1}$. Since $w^{-1} = \frac{1}{r}e^{-i\theta}$, it is clear that $\arg w^{-1} = -\arg w$, proving the claim. $\square$

$\boxed{\text{Example}}$

(1) The principal argument of $z = (\sqrt{3} - i)^6$. An argument of $\sqrt{3} - i$ is $-\pi/6$. By the proposition

$$\arg(\sqrt{3} - i)^6 = 6 \arg \sqrt{3} - i = 6(-\pi/6) = -\pi.$$

But this isn't the principal argument of $(\sqrt{3} - i)^6$ since it doesnt lie in the interval $(-\pi, \pi]$. So $\mathrm{Arg}(\sqrt{3}-i)^6 = \pi$.

(2) The proposition is not true when arg is replaced by Arg. A counterexample is $z = i$ and $w = -1$. Then $\mathrm{Arg}\, z = \pi/2$, $\mathrm{Arg}\, w = \pi$, but $\mathrm{Arg}\, zw = \mathrm{Arg}\, {-i} = -\pi/2$. But $\mathrm{Arg}\, z + \mathrm{Arg}\, w = 3\pi/2$. //

Roots of Complex Numbers

Lemma Two nonzero complex numbers z, w are equal if and only if $|z| = |w|$ and $\arg z = \arg w$.

Definition (Roots) Let w be a nonzero complex number. An n^{th} root of w is a solution to $z^n = w$.

The set of all nth roots of w is denoted by $w^{1/n}$. The symbol $\sqrt[n]{}$ is reserved to denote the unique positive nth root of a positive real number. //

Proposition (Distinct roots) There are precisely n distinct nth roots of w. Namely,

$$c_k = \sqrt[n]{|w|}\, e^{i\left(\frac{\mathrm{Arg}\, w}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

Proof. Let $z = re^{i\theta}$ and $w = |w| e^{i \mathrm{Arg}\, w}$. We solve

$$r^n e^{in\theta} = z^n = w = |w| e^{i \mathrm{Arg}\, w}.$$

By the lemma, these are equal iff $r^n = |w|$ and $n\theta = \mathrm{Arg}\, w + 2k\pi$ where k is any integer. Hence,

$$z = re^{i\theta} = \sqrt[n]{|w|}\, e^{i\left(\frac{\mathrm{Arg}\, w}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbb{Z}.$$

We obtain all unique n^{th} roots by taking $k = 0, 1, \dots, n-1$ since
$$\sum_{k=0}^{n-1} \frac{2k\pi}{n} = 2\pi.$$

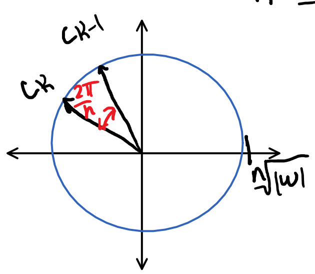
With the notation of the proposition, the principal root of w is
$$c_0 = \sqrt[n]{|w|}\, e^{i\frac{\operatorname{Arg} w}{n}}.$$

If we introduce the notation $\omega_n = e^{i\frac{2\pi}{n}}$, then we can write
$$\omega_n^{k} = e^{i\frac{2k\pi}{n}}$$

According to the proposition, the complex numbers
$$1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}$$
(c.f. Problem Set 1 p3)

are the distinct solutions to $z^n = 1$, the nth roots of unity.

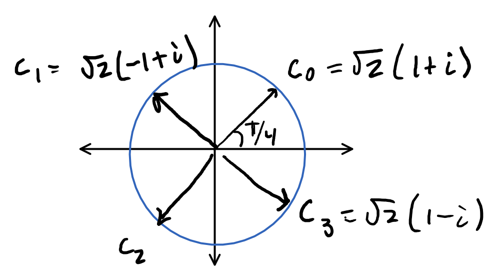
We can always write the roots of w in terms of the principal root and the roots of unity:
$$c_k = \underbrace{\sqrt[n]{|w|}}_{c_0}\, e^{i\left(\frac{\operatorname{Arg} w}{n} + \frac{2k\pi}{n}\right)} = \underbrace{\sqrt[n]{|w|}\, e^{i\frac{\operatorname{Arg} w}{n}}}\ \underbrace{e^{\frac{2k\pi}{n}}}_{\omega_n^k}$$



Example

(1) We compute explicitly the 4^{th} roots of -16.
$$c_k = \sqrt[4]{16}\; e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$$
$$= 2\, e^{i\pi/4}\, e^{\frac{k\pi}{2}}$$

So $c_0 = 2e^{i\pi/4} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(1+i).$

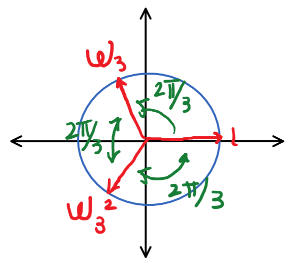


Then $c_1 = \sqrt{2}(-1+i)$, $c_2 = \sqrt{2}(-1-i)$, $c_3 = \sqrt{2}(1-i)$

(2) We compute explicitly the 3^{rd} roots of unity.

The third roots of unity are

$$1,\ \omega_3,\ \omega_3^2$$



where

$$\omega_3 = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Then

$$\omega_2 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

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Basic Topology of $\mathbb{C}$

The final topic of Ch 1 is an introduction to the basic topological ideas. The purpose is to define the kind of subsets of $\mathbb{C}$ that are suitable for doing complex analysis, namely:

nonempty open connected sets.

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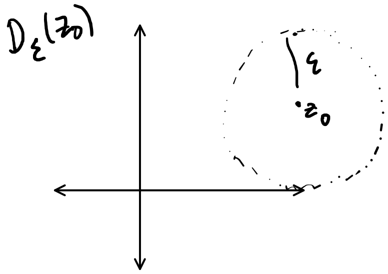
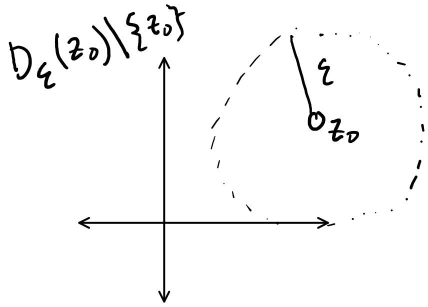
Definition (open disk / neighborhood) Let $\varepsilon > 0$. The open disk (of radius ε centered at z_0) is the set

$$D_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}.$$

We may refer to such an open disk as a neighborhood or ε-neighborhood of z_0. A deleted open disk / neighborhood is a set of the form

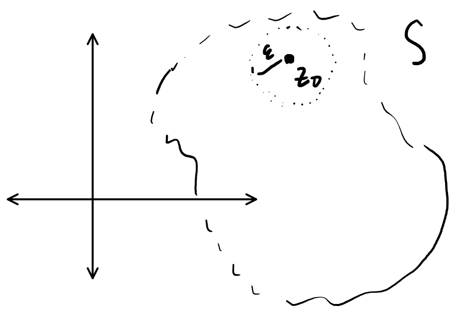
$$D_\varepsilon(z_0) \setminus \{z_0\}$$





Points within the same ε-neighborhood are "close" in the sense that they are within a distance of 2ε from eachother.

$\boxed{\text{Definition (Interior Point)}}$ Let $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is an interior point of S if $\exists\, \varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq S$



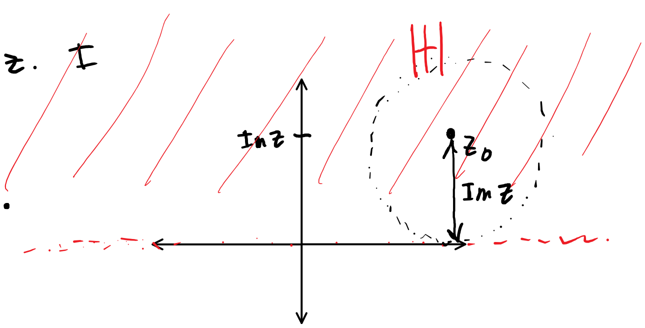
$\boxed{\text{Example}}$ The open upper half-plane

$$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$$

consists entirely of interior points.

Proof. Let $z \in \mathbb{H}$. Then $\operatorname{Im} z > 0$. Put $\varepsilon = \operatorname{Im} z$. I need to show

$$D_\varepsilon(z) \subseteq \mathbb{H}.$$



Let $w \in D_\varepsilon(\ )$. Then $|w - z| < \varepsilon = \operatorname{Im} z$.

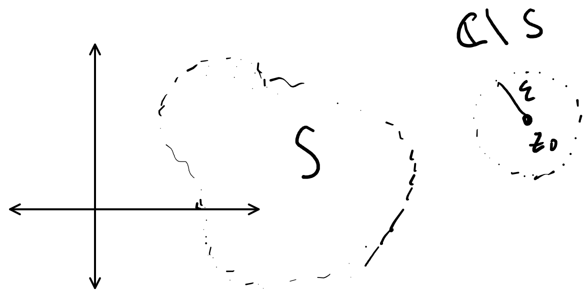
$$\operatorname{Im} z > |w - z| \geq |\operatorname{Im}(w - z)|$$
$$= |\operatorname{Im} w - \operatorname{Im} z|$$

This says

$$-\operatorname{Im} z < \operatorname{Im} w - \operatorname{Im} z < \operatorname{Im} z$$
$$\Rightarrow \quad 0 < \operatorname{Im} w. \quad \text{So } w \in \mathbb{H}.$$

$\blacksquare$

Definition (Exterior Point) Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C} \setminus S$ (the complement of S) is an exterior point if it is interior to $\mathbb{C} \setminus S = \{z_0 \in \mathbb{C},\ z_0 \notin S\}$.



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Example The points exterior to $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are the points z such that $\operatorname{Im} z < 0$.

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Definition (Boundary Point) Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is a boundary point of S if it is not an interior point to S and is not an exterior point to S. That is, every open disk contains points in S and not in S.

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